Research report 168

A SEQUENT SYSTEM FOR LEWIS'S
COUNTERFACTUAL LOGIC VC

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(RR168)

This report presents a technical result concerning practical proof systems for counterfactual (also known as conditional} logics. In a 1983 paper, H.C.M. de Swart gave sequent based proof systems for two counterfactual logics; Stalnaker's VCS and Lewis's VC. In this report I demonstrate that de Swart's system for VC is incorrect by giving a counterexample. This counterexample does not effect de Swart's system for VCS. Then I give a new sequent based proof system for VC together with soundness and completeness proofs. The system I give is closely modelled on de Swart's.

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§1 Introduction

Despite containing a section called "Introduction", this report is in no way introductory. Instead, this report presents a technical result on proof systems for counterfactual logics (also known as conditional logics); namely, I first point out that an extant proof system for one particular counterfactual logic is incorrect, and then give and prove correct a new proof system for that logic.

Perhaps the best known counterfactual logics is David Lewis's (1973) "official" counterfactual logic VC. H C M de Swart (1983) presented first a sequent based proof system for the very closely related logic VCS (in fact Robert Stalnaker's (1968) counterfactual logic), together with soundness and completeness proofs, and then a proof system for VC. Unfortunately, the soundness and completeness proofs for VC were only sketched. In this report I show that de Swart's system for VC is incorrect, in that there is a theorem of VC which the system reports to be a non-theorem. This report concentrates exclusively on VC; de Swart's work on VCS is not affected by the counterexample to VC.

In the rest of this section, I very briefly introduce Lewis's logic VC. In §2 I describe de Swart's system for VC and in §3 I give a counterexample to this system. In §4 I give a sequent based proof system for VC, for which I give soundness and completeness proofs in §5 and §6 respectively.

The language of VC contains standard propositional connectives $\wedge$, $\vee$, $\neg$, $\supset$, the propositional constants $\top$ and $\bot$, and the extra connectives $\leq$ and $\Box\!\!\rightarrow$. $A \leq B$ is read as "A is at least as possible as B". The connective $\Box\!\!\rightarrow$ is used for counterfactual implication; $A \Box\!\!\rightarrow B$ is read as "If A were the case, then B would be the case". The following definition gives the semantics of VC.

Lewis defines $\Box\!\!\rightarrow$ and $\leq$ in terms of each other. In view of this interdefinedness, it is only necessary to consider one of the connectives. In this report $\leq$ is used.

Definition 1.1

A *model for VC* is a quadruple $< I, R, \leq, [\ ] >$ which satisfies:

(1) I is a nonempty set of possible worlds.

(2) R is a binary relation on I, representing the mutual accessibility relation of possible worlds.

(3) $\leq$ is a three place relation on I, s.t. for each $i \in I$ there is a binary relation $\leq_i$ on I. Furthermore, each $\leq_i$ must be transitive and connected on $\{ j \mid j \in I \text{ and } iRj \}$. (The latter requirement is that if iRj and iRk then either $j \leq_i k$ or $k \leq_i j$ or both must be true).

(4) $[\ ]$ assigns to each formula A of VC a subset $[A]$ of I, representing the set of worlds where A is true. $[\ ]$ must satisfy the following requirements:

(4.1) $[A \wedge B] = [A] \cap [B]$

(4.2) $[A \vee B] = [A] \cup [B]$

(4.3) $[\neg A] = I - [A]$

(4.4) $[A \supset B] = (I - [A]) \cup [B]$

(4.5) $[\top] = I$

(4.6) $[\bot] = \varnothing$

(4.7) $[A \leq B] = \{ i \mid i \in I \text{ and for all } j \in [B] \text{ s.t. } iRj \text{ there is some } k \in [A] \text{ s.t. } k \leq_i j \}$

(5) *(The Centering Assumption)*
R is reflexive on I; and if iRj and $i \neq j$ then $\neg\, j \leq_i i$, (and so by the connectivity of $\leq_i$ and reflexivity of R, and in an obvious notation, $i <_i j$).

The connective $\Box\!\!\rightarrow$ can be defined in terms of $\leq$ by

$$A \Box\!\!\rightarrow B \equiv_{df} (\bot \leq A) \vee \neg((A \wedge \neg B) \leq (A \wedge B))$$

and its semantics are given by

$[A \Box\!\!\rightarrow B] = \{\, i \mid i \in I$ and if there is some $j \in [A]$ s.t. iRj then there is some $k \in [A \wedge B]$ s.t. there is no $l \in [A \wedge \neg B]$ s.t. iRl and $l \leq_i k \,\}$

Definition 1.2

A formula A is a *theorem of VC*, written "$\models_{VC}$ A", if and only if, in every model $< I, R, \leq, [\;] >$ for VC, $[A] = I$.

§2 de Swart's System for VC

Definition 2.1

A *signed formula* is any formula of the form TA or FA where A is a formula of VC.

Definition 2.2

A *sequent* is a set of signed formulas. Below, I will sometimes write "S, TA" to mean "$S \cup \{TA\}$", and similarly with more than one signed formula.

Each rule that is defined below consists of one sequent above one or more other sequents derived from it.

Definition 2.3

A *derivation* for a finite *sequent* S is a finite schema of sequents such that

(a) S is the highest sequent in the schema.

(b) If a sequent in the schema has any sequents immediately below it, they are the sequents derived from S by applying one of the rules.

(c) If a sequent has no sequents below it, then $T\bot \in S$ or $F\top \in S$ or for some formula B, $TB \in S$ and $FB \in S$.

Definition 2.4

A *derivation* for a *formula* A of VC is a derivation of the sequent { FA }.

The rules are as follows.

T$\wedge$	S, TB$\wedge$C S, TB, TC	F$\wedge$	S, FB$\wedge$C S, FB | S, FC
T$\vee$	S, TB$\vee$C S, TB | S, TC	F$\vee$	S, FB$\vee$C S, FB, FC
T$\supset$	S, TB$\supset$C S, FB | S, TC	F$\supset$	S, FB$\supset$C S, TB, FC
T$\neg$	S, T$\neg$B S, FB	F$\neg$	S, F$\neg$B S, TB
T$\leq$	S, TB$\leq$C S, TB$\leq$C, TB | S, TB$\leq$C, FC	F$\leq$	S, FB$\leq$C S, FB$\leq$C, FB

There is one additional rule that is considerably more complicated. Its general name is F$\leq$(m,n). It applies to a set of m formulas of the form FA$\leq$D and n formulas of the form TU$\leq$V. It is only applicable if m$\geq$1, but n may be 0.

[de Swart 1983] only gives special cases of this rule and the reader is left to infer the general case. Fortunately, the counterexample in §3 relies only on the special cases which were explicitly given.

F$\leq$(1,0)	S, FA$\leq$D FA , TD
F$\leq$(1,1)	S, FA$\leq$D, $TU_1\leq V_1$ FA , TD , FV_1 | FA , TU_1
F$\leq$(1,2)	S, FA$\leq$D, $TU_1\leq V_1$,$TU_2\leq V_2$ FA , TD , FV_1, FV_2 | FA , TU_1, FV_2 | FA , FV_1 , TU_2 | †(1,2)
F$\leq$(1,3)	S, FA$\leq$D, $TU_1\leq V_1$, $TU_2\leq V_2$, $TU_3\leq V_3$ FA,TD,FV_1,FV_2,FV_3 | FA,TU_1,FV_2,FV_3 | FA,FV_1,TU_2,FV_3 | FA,FV_1,FV_2,TU_3 | †(1,3)

†(1,2) F$\leq$(1,2) leads to a derivation if each of the first three sequents above is derivable and either of the following sequents (or both) is derivable.

FA, TU_1, TV_2

FA, TU_2, TV_1

$\dagger(1,3)$ $F\leq(1,3)$ leads to a derivation if each of the first four sequents above is derivable and each of the sequents is derivable in one (or more) of the following six sets of sequents.

$$FA,\ TU_1,\ TV_2 \mid FA,\ TU_1,\ TV_3 \mid FA,\ TU_2,\ TV_3$$
$$FA,\ TU_1,\ TV_2 \mid FA,\ TU_1,\ TV_3 \mid FA,\ TU_3,\ TV_2$$
$$FA,\ TU_2,\ TV_1 \mid FA,\ TU_2,\ TV_3 \mid FA,\ TU_1,\ TV_3$$
$$FA,\ TU_2,\ TV_1 \mid FA,\ TU_2,\ TV_3 \mid FA,\ TU_3,\ TV_1$$
$$FA,\ TU_3,\ TV_1 \mid FA,\ TU_3,\ TV_2 \mid FA,\ TU_1,\ TV_2$$
$$FA,\ TU_3,\ TV_1 \mid FA,\ TU_3,\ TV_2 \mid FA,\ TU_2,\ TV_1$$

Note that in the rule $F\leq(m,n)$, the sequent S disappears in the derived sequents. This is because the derived sequents can be seen as referring to different possible worlds and so many statements about the original world become irrelevant.

Remark 2.5

Note that, due to the above definition of the six sets of three sequents, $F\leq(1,3)$ can lead to a derivation only if:

for any pair (j,k) with $1 \leq j < k \leq 3$, either $\{FA,\ TU_j,\ TV_k\}$ or $\{FA,\ TU_k,\ TV_j\}$ is derivable.

This remark will be used in the proof of Theorem 3.2 in the next section.

§3 A Counterexample to de Swart's System

Consider the formula of VC

$$(A \leq C \wedge C \leq D \wedge D \leq (\neg D \wedge B)) \supset (A \leq B) \qquad (1)$$

Theorem 3.1

(1) is a theorem of VC.

Proof

It is necessary to prove that $[\ (1)\ ] = I$ in any model $< I, R, \leq, [\ ] >$ for VC. So from the semantics of VC, it is necessary to prove

$$(I - ([A \leq C] \cap [C \leq D] \cap [D \leq (\neg D \wedge B)])) \cup [A \leq B] = I$$

To do this, it is sufficient to prove

For any $i \in I$, if $i \in ([A \leq C] \cap [C \leq D] \cap [D \leq (\neg D \wedge B)])$ then $i \in [A \leq B]$

Consider any $i_0 \in ([A \leq C] \cap [C \leq D] \cap [D \leq (\neg D \wedge B)])$. Choose any $i_1 \in I$ such that $i_0 R i_1$ and $i_1 \in [B]$ (if there is no such i_1 then $i_0 \in [A \leq B]$ is trivially true). Now, either $i_1 \in [D]$ or $i_1 \in [\neg D]$. In either case, I will prove that there must be some $i' \in I$ s.t. $i_0 R i'$, $i' \leq_{i_0} i_1$ and $i' \in [A]$. Since the choice of i_1 was arbitrary, I will have established $i_0 \in [A \leq B]$.

If $i_1 \in [D]$, then, since $i_0 \in [C \leq D]$, there is some $i_2 \in I$ s.t. $i_0 R i_2$, $i_2 \in [C]$ and $i_2 \leq_{i_0} i_1$. Then since $i_0 \in [A \leq C]$, there is some $i' \in I$ s.t. $i_0 R i'$, $i' \in [A]$ and $i' \leq_{i_0} i_2$. Since $\leq_{i_0}$ is transitive, $i' \leq_{i_0} i_1$.

If $i_1 \in [\neg D]$, then $i_1 \in [\neg D \wedge B]$. Since $i_0 \in [D \leq (\neg D \wedge B)]$, there is some $i_2 \in I$ s.t. $i_0 R i_2$, $i_2 \in [D]$ and $i_2 \leq_{i_0} i_1$. Then by a similar chain to that in the last paragraph we can establish that there is some $i' \in I$ with the required conditions.

□ (Theorem 3.1)

Theorem 3.2
(1) is not derivable in de Swart's System

Proof
A derivation for (1) is a derivation for

$$\{ F ((A \leq C \wedge C \leq D \wedge D \leq (\neg D \wedge B)) \supset A \leq B) \} \tag{2}$$

In searching a derivation, we find one sequent which only $F\leq(m,n)$ applies to and which does not contain FE and TE for any E. This sequent is derived as follows. (Each sequent is annotated by the rule used to produce it, and "RHS" indicates that the right hand sequent of the two possible new sequents was chosen.)

$\{ F (A \leq C \wedge C \leq D \wedge D \leq (\neg D \wedge B)) \supset A \leq B) \}$	(2)
$\{ T (A \leq C \wedge C \leq D \wedge D \leq (\neg D \wedge B))), FA \leq B \}$	$F\supset$
$\{ TA \leq C, TC \leq D, TD \leq (\neg D \wedge B), FA \leq B \}$	$T\wedge \times 2$
$\{ TA \leq C, TC \leq D, TD \leq (\neg D \wedge B), FA \leq B, FA \}$	$F\leq$
$\{ TA \leq C, TC \leq D, TD \leq (\neg D \wedge B), FA \leq B, FA, FC, FD, F(\neg D \wedge B) \}$	$T\leq$ (RHS) $\times 3$
$\{ TA \leq C, TC \leq D, TD \leq (\neg D \wedge B), FA \leq B, FA, FC, FD, FB \}$	$F\wedge$ (RHS)

To substantiate Theorem 3.2 it remains only to show that for each possible application of a $F\leq(m,n)$ rule to the last sequent above, either one of the sequents $S_1, \ldots, S_{m+n}$ is not derivable or the special case $\dagger(m,n)$ fails.

$F\leq(1,0)$ applied to $FA \leq B$.

This leads to the single sequent $S_1 = \{ FA, TB \}$, which is not derivable.

$F\leq(1,1)$ applied to $FA \leq B$, $TA \leq C$.

One of the derived sequents is $S_1 = \{ FA, TB, FC \}$ which is not derivable.

$F\leq(1,1)$ applied to $FA \leq B$, $TC \leq D$.

One of the derived sequents is $S_1 = \{ FA, TB, FD \}$ which is not derivable.

$F\leq(1,1)$ applied to $FA \leq B$, $TD \leq (\neg D \wedge B)$

One of the derived sequents is $S_1 = \{ FA, TB, F(\neg D \wedge B) \}$ which is not derivable.

$F\leq(1,2)$ applied to $FA \leq B$, $TA \leq C$, $TC \leq D$.

One of the derived sequents is $S_1 = \{ FA, TB, FC, FD \}$ which is not derivable.

$F\leq(1,2)$ applied to $FA \leq B$, $TA \leq C$, $TD \leq (\neg D \wedge B)$

One of the derived sequents is $S_1 = \{ FA, TB, FC, F(\neg D \wedge B) \}$ which is not derivable.

$F\leq(1,2)$ applied to $FA \leq B$, $TC \leq D$, $TD \leq (\neg D \wedge B)$

One of the derived sequents is $S_2 = \{ FA, TC, F(\neg D \wedge B) \}$ which is not derivable.

$F\leq(1,3)$ applied to $FA \leq B$, $TA \leq C$, $TC \leq D$, $TD \leq (\neg D \wedge B)$.

Here, all the sequents $S_1, \ldots, S_4$ are derivable, but the special rule $\dagger(1,3)$ fails. To see this, recall Remark 2.5, and apply it to $TC \leq D$ and $TD \leq (\neg D \wedge B)$. Then the relevant sequents are $\{ FA, TC, T(\neg D \wedge B) \}$ and $\{ FA, TD, TD \}$, neither of which is derivable.

□ (Theorem 3.2)

Together, Theorems 3.1 & 3.2 establish that de Swart's system for VC is incorrect.[1]

[1] The reader may be interested in how this counterexample was discovered. Curiously, I wrote down the system of §4 before I realised that de Swart's system was wrong. In studying the relationship between the two systems, I was able to

§4 A Sequent Based Proof System for VC

In this section I describe a proof system for VC similar to de Swart's. In §5 I prove the soundness theorem for this system, and in §6 I prove the completeness theorem.

All the definitions of §2 apply equally to this system, and all the rules are the same except for F≤(m,n). The definition of F≤(m,n) is given below.

$$\text{F}{\le}(m,n) \qquad \frac{S,\ FA_1{\le}D_1, \ldots, FA_m{\le}D_m, TU_1{\le}V_1, \ldots, TU_n{\le}V_n}{S_1 \mid S_2 \mid \cdots \mid S_m \mid (*)}$$

where

$$S_i = \{FA_1, \ldots, FA_m, TD_i, FV_1, \ldots, FV_n\} \text{ for } 1 \le i \le m$$

and where (*) is the following special condition, which only applies if $n \ge 1$.

(*) There is a sequence $i_1, i_2, \ldots, i_n$ which is a permutation of $1, 2, \ldots, n$ and is such that each of the following sequents is derivable[2].

$$\{FA_1, \ldots, FA_m, TU_{i_1}\}$$

$$\{FA_1, \ldots, FA_m, TU_{i_2}, FV_{i_1}\}$$

$$\{FA_1, \ldots, FA_m, TU_{i_3}, FV_{i_1}, FV_{i_2}\}$$

$$\ldots$$

$$\{FA_1, \ldots, FA_m, TU_{i_n}, FV_{i_1}, FV_{i_2}, \ldots, FV_{i_{n-1}}\}$$

In the rest of this report, references to the proof system for VC refer to this system, and not de Swart's.

§5 Soundness

Theorem 5.1 (Soundness Theorem)

For any formula A of VC, if there is a derivation for A then $\models_{VC}$ A.

Proof

If there is a derivation for A, there is a derivation with FA as its uppermost sequent. So it suffices to show that for any sequent $\{TB_1, TB_2, \ldots, TB_m, FC_1, \ldots, FC_n\}$ in the derivation of FA,

$$\models_{VC}\ (B_1 \wedge B_2 \wedge \cdots \wedge B_m) \supset (C_1 \vee \cdots \vee C_n) \qquad (3)$$

This will be done by induction, on the size of the derivation.

Induction Base

A derivation of minimal size is one in which the original sequent contains $T\bot$, $F\top$, or both TB and FB for some formula B.

construct the above counterexample to their equivalence. It was only then that I noticed that de Swart's system gives the wrong answer in this case.

[2] This condition could be expressed as a complicated condition on various sets of sequents, mirroring the use of †(1,2) and †(1,3) in the presentation of de Swart's system. However, this method of presentation makes the general rule much clearer.

In the first case, we need $\models_{VC} (\perp \wedge B_1 \cdots \wedge B_m) \supset (C_1 \vee \cdots \vee C_n)$, which is certainly satisfied by VC. In the second and third cases, we need $\models_{VC} (B_1 \wedge B_2 \wedge \cdots \wedge B_m) \supset (\top \vee C_1 \vee \cdots \vee C_n)$ and $\models_{VC} (B \wedge B_1 \wedge \cdots \wedge B_m) \supset (B \vee C_1 \vee \cdots \vee C_n)$ respectively, both of which are satisfied by VC.

Induction Step

For each rule, we have to show that if (3) holds for all the sequents the rule leads to, then (3) holds for the original sequent. I will give an example of doing this for one of the propositional rules, which are straightforward, and then I shall do this for the rules T≤, F≤, and F≤(m,n). This will complete the proof of the soundness theorem.

Induction Step for the rule T⊃

The induction hypothesis is that, in any model $< I, R, \leq, [\,] >$,

$$[(B_1 \wedge \cdots \wedge B_m) \supset (C_1 \vee \cdots \vee C_n \vee D)] = I \tag{4}$$

and

$$[(B_1 \wedge \cdots \wedge B_m \wedge E) \supset (C_1 \vee \cdots \vee C_n)] = I \tag{5}$$

We have to show that

$$[(B_1 \wedge \cdots \wedge B_m \wedge (D \supset E)) \supset (C_1 \vee \cdots \vee C_n)] = I$$

Consider any $i \in I$. It is sufficient to consider the case where

$$i \in [B_1],\ i \in [B_2], \ldots,\ i \in [B_m] \tag{6}$$

$$i \in [D \supset E] \tag{7}$$

and to show

$$i \in [C_1 \vee \cdots \vee C_n] \tag{8}$$

Now (7) means that either $i \notin [D]$ or $i \in [E]$. If $i \notin [D]$ then, from (4) and (6), we have (8) as required. If $i \in [E]$ then, from (5) and (6), we have (8) as required.
□ (induction step for T⊃)

Induction Step for the Rule T≤

The induction hypothesis is that, in any model $< I, R, \leq, [\,] >$,

$$[(B_1 \wedge \cdots \wedge B_m \wedge (U \leq V) \wedge U) \supset (C_1 \vee \cdots \vee C_n)] = I \tag{9}$$

and

$$[(B_1 \wedge \cdots \wedge B_m \wedge (U \leq V)) \supset (C_1 \vee \cdots \vee C_n \vee V)] = I \tag{10}$$

We need to show that

$$[(B_1 \wedge \cdots \wedge B_m \wedge (U \leq V)) \supset (C_1 \vee \cdots \vee C_n)] = I$$

Consider any $i \in I$. It is sufficient to consider only the case where

$$i \in [B_1],\ i \in [B_2], \ldots,\ i \in [B_m] \tag{11}$$

$$i \in [U \leq V] \tag{12}$$

and to show

$$i \in [C_1 \vee \cdots \vee C_n] \tag{13}$$

Suppose $i \notin [V]$. Then from (10), (11) and (12), we get (13) as required.
Suppose $i \in [V]$. From (12) and the semantics of VC, we have

$$\text{There is some } k \in [U] \text{ s.t. } iRk \text{ and } k \leq_i i \tag{14}$$

But, by the Centering Assumption, if $k \leq_i i$ then $k = i$, so from (14) $i \in [U]$. Using (9), (11) and (12), we get (13) as required.
□ (induction step for T≤)

Induction step for F≤
The induction hypothesis is

$$[(B_1 \wedge \cdots \wedge B_m) \supset (C_1 \vee \cdots \vee C_n \vee (A \leq B) \vee A)] = I \tag{15}$$

We need to show that

$$[(B_1 \wedge \cdots \wedge B_m) \supset (C_1 \vee \cdots \vee C_n \vee (A \leq B))] = I$$

Consider any $i \in I$. It is sufficient to consider only the case where

$$i \in [B_1],\ i \in [B_2], \ldots,\ i \in [B_m] \tag{16}$$

$$i \notin [C_1],\ i \notin [C_2], \ldots,\ i \notin [C_n] \tag{17}$$

and to show

$$i \in [A \leq B] \tag{18}$$

Suppose $i \notin [A]$. Then by (15), (16), (17), we have (18) as required.
Suppose $i \in [A]$. Now, by the Centering Assumption, if $j \in I$ and iRj then $i \leq_i j$. So in particular, if $j \in [B]$ and iRj then $i \leq_i j$. This gives us (18) as required.
□ (induction step for F≤)

Induction step for F≤(m,n), m≥1, n≥0
The induction hypothesis is that, for any model $< I, R, \leq, [\ ]>$,

$$[D_j \supset (A_1 \vee \cdots \vee A_m \vee V_1 \vee \cdots \vee V_n)] = I \quad \text{for } 1 \leq j \leq m \tag{19}$$

and, assuming without loss of generality that $j_1=1, j_2=2, \ldots, j_n=n$,

$$[U_1 \supset A_1 \vee \cdots \vee A_m] = I \tag{20.1}$$

$$[U_2 \supset A_1 \vee \cdots \vee A_m \vee V_1] = I \tag{20.2}$$

$$[U_3 \supset A_1 \vee \cdots \vee A_m \vee V_1 \vee V_2] = I \tag{20.3}$$

$$\cdots$$

$$[U_n \supset A_1 \vee \cdots \vee A_m \vee V_1 \vee V_2 \vee \cdots \vee V_{n-1}] = I \tag{20.n}$$

We need to show, given (19) and (20),

$$[((U_1 \leq V_1) \wedge \cdots \wedge (U_n \leq V_n)) \supset ((A_1 \leq D_1) \vee \cdots \vee (A_m \leq D_m))] = I$$

Consider any $i \in I$. It is sufficient to consider only those $i \in I$ such that

$$i \in [U_1 \leq V_1], \ldots, i \in [U_n \leq V_n] \tag{21}$$

$$i \notin [A_2 \leq D_2], \ldots, i \notin [A_m \leq D_m] \tag{22}$$

and to show that

$$i \in [A_1 \leq D_1]$$

Since, if $\forall i_1 \in I\ iRi_1 \supset i_1 \notin [D_1]$, it follows trivially that $i \in [A_1 \leq D_1]$, we may assume that

$$\exists\, i_1 \in I \text{ s.t. } iRi_1 \text{ and } i_1 \in [D_1] \tag{23}$$

It now is sufficient to show, given (19), (20), (21), (22) and (23), that

$$\exists\, i_0 \in I \text{ s.t. } iRi_0,\ i_0 \leq_i i_1 \text{ and } i_0 \in [A_1] \tag{24}$$

Proof (Induction step for F≤(m,n))
I will demonstrate the existence of a sequence of worlds

$$\begin{array}{llll} i_1 = & i_{1,0}, & i_{1,1}, \ldots & i_{1,n_1}, \\ & i_{2,0}, & i_{2,1}, \ldots & i_{2,n_2}, \\ & \cdot & \cdot & \cdot \\ & \cdot & \cdot & \cdot \\ & i_{k,0}, & i_{k,1}, \ldots & i_{k,n_k} \end{array}$$

Each new world in the sequence will be $\leq_i$ the previous one, and so by transitivity $\leq_i i_1$. It will turn out that i_{k,n_k} will satisfy the requirements for i_0 in (24).

Set $i_{1,0}=i_1$. From (19) and (23) we have $i_{1,0} \in [A_1 \vee \cdots \vee A_m \vee V_1 \vee \cdots \vee V_n]$.
Now suppose, in general, that we have

$$i_{f,g} \in [A_1 \vee \cdots \vee A_m \vee V_1 \vee \cdots \vee V_{h_g}] \tag{25.a}$$

$$i_{f,g} \leq_i i_{f',g'} \text{ for all } f' \leq f \text{ and } g' \leq g \tag{25.b}$$

If $i_{f,g} \in [A_1 \vee \cdots \vee A_m]$, then set $n_f = g$.
Otherwise, $i_{f,g} \in [V_1 \vee \cdots \vee V_{h_g}]$. So, for some h with $1 \leq h \leq h_g$, $i_{f,g} \in [V_h]$. But, from (21), $i \in [U_h \leq V_h]$, so there is some $i_{f,g+1}$ s.t. $iRi_{f,g+1}$ and $i_{f,g+1} \leq_i i_{f,g}$ and $i_{f,g+1} \in [U_h]$.
Because $i_{f,g+1} \leq_i i_{f,g}$, and from the transitivity of $\leq_i$, we get (25.b).
From (20.h), and setting $h_{g+1} = h-1$, $i_{f,g+1} \in [A_1 \vee \cdots \vee A_m \vee V_1 \vee \cdots \vee V_{h-1}]$, satisfying (25.a).
Since $1 \leq h \leq h_g$, we have $0 \leq h_{g+1} < h_g$. So $h_1, h_2, \ldots,$ is a strictly decreasing sequence of integers bounded below by 0. Thus it must be a finite sequence, and by construction it must end with some $h_{n_f}=0$.
Thus, given (25) we can show that there is a sequence of worlds $i_{f,g}, i_{f,g+1}, \ldots, i_{f,n_f}$ such that $i_{f,n_f} \in [A_1 \vee \cdots \vee A_m]$. The sequence starts because setting f=1, g=0 and $h_g=n$ satisfies the conditions of (25).
Now suppose, in general, that

$$i_{f,n_f} \in [A_1 \vee \cdots \vee A_m] \tag{26.a}$$

$$i_{f,n_f} \leq_i i_1 \text{ and } i_{f,n_f} \leq_i i_{f',n_{f'}} \text{ for all } f' \leq f \tag{26.b}$$

So $i_{f,n_f} \in [A_{a_f}]$ for some a_f.
If $a_f = 1$, then set k=f and $i_0 = i_{k,n_k}$.
Otherwise, we have from (23), $i \notin [A_{a_f} \leq D_{a_f}]$. So there is some $i_{f+1,0} \in I$ s.t. $i_{f+1,0} \in [D_{a_f}]$ and $iRi_{f+1,0}$ and $i_{f+1,0} <_i i_{f,n_f}$ and

$$\forall j \in I\ (iRj \text{ and } j \leq_i i_{f+1,0}) \supset j \notin [A_{a_f}] \tag{27}$$

By the transitivity of $\leq_i$, and the fact that $i_{f+1,0} <_i i_{f,n_f}$, $i_{f+1,0}$ satisfies (25.b). By (19), $i_{f+1,0} \in [A_1 \vee \cdots \vee A_m \vee V_1 \vee \cdots \vee V_n]$, so $i_{f+1,0}$ satisfies (25.a). Therefore $i_{f+1,n_{f+1}}$ satisfies (26.a) and (26.b).

This establishes a sequence $i_{1,n_1}, i_{2,n_2}, \ldots$ and an associated sequence $a_1, a_2, \ldots$ s.t. $i_{f,n_f} \in [A_{a_f}]$.

Now, the latter sequence cannot have any repetitions, for if g>f then, by (25) and (26), $i_{g,n_g} \leq_i i_{f+1,0}$, and so by (27) $i_{g,n_g} \notin [A_{a_f}]$.

So the sequence $a_1, a_2, \ldots$ is at most length m. Yet it can only stop when $a_k=1$. So for some finite k, $a_k=1$, and we have $i_{f_k,n_{f_k}} \in [A_1]$ and $i_{f_k,n_{f_k}} \leq_i i_1$, and so we have satisfied (24).

□ (induction step for F≤(m,n) and Soundness Theorem)

§6 Completeness

Definition 6.1

A *Hintikka element* is a finite set S of signed formulas such that:

if $TB \wedge C \in S$ then $TB \in S$ and $TC \in S$; and
if $FB \wedge C \in S$ then $FB \in S$ or $FC \in S$; and
if $TB \vee C \in S$ then $TB \in S$ or $TC \in S$; and
if $FB \vee C \in S$ then $FB \in S$ and $FC \in S$; and
if $TB \supset C \in S$ then $FB \in S$ or $TC \in S$; and
if $FB \supset C \in S$ then $TB \in S$ and $FC \in S$; and
if $T\neg B \in S$ then $FB \in S$; and
if $F\neg B \in S$ then $TB \in S$.

A *VC-Hintikka element* is a Hintikka element S that also satisfies

if $TB \leq C \in S$ then $TB \in S$ or $FC \in S$; and
if $FB \leq C \in S$ then $FB \in S$.

Theorem 6.2 (Completeness Theorem)

For any formula A of VC, if there is no derivation for A then $\not\models_{VC} A$.

Proof

It is sufficient to prove that,

(28) for any sequent $S = \{ TB_1, \ldots, TB_n, FC_1, \ldots, FC_m \}$ which is not derivable, there is a model $<I, R, \leq, [\,]>$ of VC and some $i \in I$ such that $i \in [\, B_1 \wedge \cdots \wedge B_n \wedge \neg C_1 \wedge \cdots \wedge \neg C_m \,]$. (I will sometimes abuse notation by writing "$i \in [\, S \,]$" for this).

Suppose that a sequent S_1 is one of the sequents derived from some sequent S_0 by one of the rules T∧, F∧, T∨, F∨, T⊃, F⊃, T¬, F¬, T≤, or F≤. Then it is easy to show that if $i \in [S_1]$ then $i \in [S_0]$ and so $i \in [S_0 \cup S_1]$. If S_0 is not derivable, then any sequence of applications of these rules must end (by a simple complexity argument) and leave at least one underivable sequent. Since the series of applications of rules has ended, none of the rules except F≤(m,n) can be applicable to the underivable sequent. Now consider the union of all the ancestor sequents of this underivable sequent in the attempted derivation. Since the definition of a VC-Hintikka element matches the definition of the rules, this union must be a VC-Hintikka element. So if S_0 is not derivable, there is some VC-Hintikka element S' which is not derivable, and such that if $i \in [S']$ then $i \in [S_0]$.

The previous paragraph shows that it is sufficient to prove (28) only for sequents S' which are VC-Hintikka elements. This will be done by induction on the maximum nesting of the symbol "≤" in S'.

Induction Base

Since S' is not derivable, S' does not contain $F\top$, $T\bot$, or FB and TB for any B.

Define a model M for VC by

$I = \{i_0\}$ which makes R and $\leq$ trivial.

For atomic propositions P, $i_0 \in [P]$ if and only if $TP \in S'$.

For non atomic formulae A of VC, [A] is defined by the above and the semantics of VC. ([] is well-defined because for each compound formula A, [A] is defined in terms of strictly simpler formulae).

It is now easy to show (28), by induction on the total size of the formulae in the sequent S'.

Induction Step

Suppose S is not derivable. Then there is a sequence of rules, not including $F\leq(m,n)$, which can be applied, to yield a VC-Hintikka element S' such that each possible application of $F\leq(m,n)$ to S' yields at least one underivable sequent.

Suppose the set of all signed formulas in S' with $\leq$ dominating is $S_1 = \{ F A_1 \leq D_1, \ldots, F A_m \leq D_m, T U_1 \leq V_1, \ldots, T U_n \leq V_n \}$.

I will show the existence of two finite sequences of sequents $S_1, \ldots S_p, \ldots$ and $S_{1,1}, \ldots S_{p,q}, \ldots$ which can be used to construct a model for S'.

If $m = 0$, the first sequence will be simply S_1 and the second will be empty.

Suppose, in general, that $S_p \subseteq S_1$ and that $S_p = \{ F A_1 \leq D_1, \ldots, F A_{m_p} \leq D_{m_p}, T U_1 \leq V_1, \ldots, T U_{n_p} \leq V_{n_p} \}$ with $m_p > 0$. We can apply $F\leq(m_p,n_p)$ to S_1 and we know by the definition of S' that the application will not close. This means that either one of the sequents $\{ F A_1, \ldots, F A_{m_p}, T D_j, F V_1, \ldots, F V_{n_p} \}$ is not derivable for some j, or the special condition (*) fails.

If the former, then set $S_{p,1} = \{ F A_1, \ldots, F A_{m_p}, T D_j, F V_1, \ldots, F V_{n_p} \}$ and set $S_{p+1} = S_p - \{ F A_j \leq D_j \}$.

If the special rule fails, then for some $k < n_p$ there must be a sequence $j_1, j_2, \ldots, j_k$ (and I will assume, without loss of generality, that it is the sequence $1, 2, \ldots, k$) with the property that: each sequent $\{ F A_1, \ldots, F A_{m_p}, T U_j, F V_1, \ldots, F V_{j-1} \}$ is derivable for $1 \leq j \leq k$, but no sequent $\{ F A_1, \ldots, F A_{m_p}, T U_j, F V_1, \ldots, F V_k \}$ is derivable for $k < j \leq n_p$. If there was no such sequence for $k < n_p$ then the special rule would not fail.

In this case, set $S_{p,q} = \{ F A_1, \ldots, F A_{m_p}, T U_{k+q}, F V_1, \ldots, F V_k \}$ for $1 \leq q \leq n_p - k$ and set $S_{p+1} = S_p - \{ T U_{k+1} \leq V_{k+1}, \ldots, T U_{n_p} \leq V_{n_p} \}$.

We can repeat this process until $m_p = 0$, when we stop. Note that if $m_p = 0$ that the first sequence ends with S_p and the second with some $S_{p-1,q}$.

Each $S_{p,q}$ is underivable, and so from each one we can build an underivable VC-Hintikka element $S'_{p,q}$, as in the opening of this completeness proof. The symbol "$\leq$" is nested strictly less deeply in each $S_{p,q}$ than in S', and none of the rules used to build $S'_{p,q}$ increases the nesting of "$\leq$". So "$\leq$" is nested strictly less deeply in each $S'_{p,q}$ than in S'. So by the induction hypothesis, and since $S_{p,q} \subseteq S'_{p,q}$, we may assume that for each $S_{p,q}$ there is a VC model $M_{p,q} = < I_{p,q}, R_{p,q}, \leq_{p,q}, [\,]_{p,q} >$ and some $i_{p,q} \in I_{p,q}$ such that $i_{p,q} \in [\, S_{p,q} \,]_{p,q}$.

Now define a VC model $< I, R, \leq, [\,] >$ containing a world i_0 as follows.

$$I = \{ i_0 \} \cup \{ i : i \in I_{p,q} \text{ for some } p,q \}$$

$$iRj \Leftrightarrow \begin{cases} i, j \in I_{p,q} \text{ and } iR_{p,q}j \\ \text{or } i = i_0 \text{ and } j = i_{p,q} \\ \text{or } i = i_0 \text{ and } j = i_0 \end{cases}$$

$i \leq_k j$ defined by the $\leq_{p,q}$ relations and by:

$i_0 \leq_{i_0} i_0$

$i_0 <_{i_0} i_{p,q}$

$i_{p,q} <_{i_0} i_{p',q'}$ if $p < p'$

$i_{p,q} \leq_{i_0} i_{p,q'}$ and $i_{p,q'} \leq_{i_0} i_{p,q}$

[ ] defined by the $[\ ]_{p,q}$ relations and by:

For atomic propositions P, $i_0 \in [P] \Leftrightarrow TP \in S'$.

For non atomic propositions Q, whether or not $i_0 \in [Q]$ is given by the above definitions and the semantics of VC.

Note that for each world in one of the models $M_{p,q}$, the above definitions leave the semantics of that world unchanged, since each such world bears exactly the same R and ≤ relations as it did in $M_{p,q}$. Also, the semantics of i_0 for atomic propositions are well defined since S′ does not contain TP and FP for any proposition P. The definition of semantics of VC defines each non-atomic proposition in terms of strictly simpler propositions, and so the semantics of all propositions in i_0 are well defined. Thus, the definition of [ ] is well defined.

It remains to show that for any formula P of VC: that if $TP \in S'$ then $i_0 \in [P]$; and that if $FP \in S'$ then $i_0 \notin [P]$. This will be done by induction on the complexity of the structure of P.

The induction base is the case of atomic formulas. In this case, these requirements are met by the definition of [ ] above.

The induction step is straightforward for signed formulas dominated by a propositional symbol. So it only remains to prove the induction step for signed formulas dominated by ≤.

(Inner) Induction step for formulas $T\,U_k \leq V_k$

If $TU_k \leq V_k \in S'$ then $i_0 \in [\,U_k \leq V_k\,]$.

Proof

Since S′ is a VC-Hintikka element, either $TU_k \in S'$ or $FV_k \in S'$. Each of these signed formulae is strictly simpler than $TU_k \leq V_k$, so by the inner induction hypothesis either $i_0 \in [\,U_k\,]$ or $i_0 \notin [V_k]$.

If $i_0 \in [\,U_k\,]$, then by the semantics of VC, we have $i_0 \in [\,U_k \leq V_k\,]$, and we have finished. So we may assume that $FV_k \in S'$ and so $i_0 \notin [V_k]$.

Suppose that for some p′, there is no $i_{p,q}$ such that $p < p'$ and $i_{p,q} \in [\,U_k\,]$. Then, from the choice of model $M_{p,q}$, for each $p < p'$, $TU_k \notin S_{p,q}$ and so, by construction, $TU_k \leq V_k \in S_p$. Then, again by construction, $FV_k \in S_{p,q}$ and so, again by the choice of $M_{p,q}$, $i_{p,q} \notin [\,V_k\,]$ for $p < p'$.

Either no $i_{p,q} \in [\,U_k\,]$ or some $i_{p,q} \in [\,U_k\,]$.

If the former, then by the above argument we have that for each p,q, $i_{p,q} \notin [\,V_k\,]$, and remembering that we assumed that $i_0 \notin [\,V_k\,]$, we have that there is no world j s.t. i_0Rj and $j \in [\,V_k\,]$. Thus $i_0 \in [\,U_k \leq V_k\,]$.

If the latter, then there is some smallest p′ s.t. for some q′ $i_{p',q'} \in [\,U_k\,]$. Then by the above argument, and the assumption that $i_0 \notin [\,V_k\,]$, we have that there is no world j s.t. i_0Rj and $j \in [\,V_k\,]$ and $j <_{i_0} i_{p',q'}$. Thus $i_0 \in [\,U_k \leq V_k\,]$.

□ (inner induction step for $TU_k \leq V_k$)

(Inner) Induction step for formulas F $U_k \leq V_k$

If $FA_k \leq D_k \in S'$ then $i_0 \notin [\ A_k \leq D_k\ ]$.

Proof

By construction, there is some S_p which does not contain any formula $FP \leq Q$ and so there is some largest p' s.t. $FA_k \leq D_k \in S_{p'}$. Then, by construction, $TD_k \in S_{p',0}$, and so by the choice of $M_{p,q}$, $i_{p',0} \in [\ D_k\ ]$.

Now consider $j \in I$ s.t. $j \leq_{i_0} i_{p',0}$. Either $j = i_0$ or $j = i_{p,q}$ for some $p \leq p'$.

If $j = i_0$, then since S' is a VC-Hintikka element, $FA_k \in S'$, and so by the (inner) induction hypothesis, $i_0 \notin [\ A_k\ ]$.

If $j = i_{p,q}$ for some $p \leq p'$, then $FA_k \leq D_k \in S_p$. Then, by construction, $FA_k \in S_{p,q}$, and so by the choice of $M_{p,q}$, $i_{p,q} \notin [\ A_k\ ]$.

So we have established that there is some p' s.t. $i_{p',0} \in [\ D_k\ ]$, and that if $j \leq_{i_0} i_{p',0}$ then $j \notin [\ A_k\ ]$. This establishes that $i_0 \notin [\ A_k \leq D_k\ ]$.

□ (inner induction step for $FA_k \leq D_k$ & Completeness Theorem)

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